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Sampled-Data System with Computation Time Delay: Optimal W-Synthesis Method

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We treat the control problem for sampled-data systems that accounts for the time delay in the computation of the control action. The problem is formulated entirely in the w domain, and the optimum design is solved using an alternative method called optimal W-synthesis method. The method has been applied to the benchmark problem of a two-mass-spring system. Design results obtained indicate that the presence of computation time delay not only reduces the relative stability and robustness but also degrades the design performance. Loss in gain and phase margins is significant if the time delay is not accounted for in the control-law synthesis. Furthermore, control activity can be highly sensitive to the inherent time delay resulting in excessive change in the control action that may eventually lead to control saturation in a real system. It is found that, when the time delay is incorporated into the control-law synthesis, stability, robustness, and performance loss because of computation time delay can be reduced.

I. Introduction

SINCE the invention of microprocessors, control system designers have found extensive use for these devices in computer-controlled feedback systems. Digital controllers have many advantages over their analog counterparts: easy reprogramming without expensive wiring changes, smaller size and lighter weight, and lower expense than analog controllers. Digital controllers, however, pose some fundamentally different characteristics that should be carefully considered by control designers, e.g., finite word length effects, quantization, conversion, and computation time delay. In this paper, only the problem associated with computation time delay will be addressed. Issues related to computation time delay have been extensively studied. Among these references, Dorato and Levis[1] showed the relevant transformation of a digital control problem into an equivalent discrete-time control problem. Mukhopadhyay[2] provided a method for designing optimum digital controllers using constrained optimization. The formulation, however, did not address the problem of computation time delay. In Ref. 3, the authors examined the computation time delay for an optimal full state-feedback regulator problem. Mita[4] showed that, in the case where the computation time delay is an integer multiple of the sampling time, the results can be derived from the conventional optimal regulator theory. Diduch and Doraiswami[5] pursued the problem of control servomechanism design and its output-feedback implementation. They considered a linear quadratic Gaussian (LQG) compensator based on an optimal estimator design. Reference 6 also gave a procedure to account for the computation time delay in a LQG compensator design structure. From this brief survey, we found that the control problem associated with computation time delay has not been extensively treated in a general low-order compensator setting.

In this paper, we examine the control design problem involving computation time delay for an arbitrary low-order digital compensator structure. The problem is defined in the w domain, and its solution is obtained using the so-called optimal W-synthesis method described in Refs. 7 and 8. This method provides a unique procedure for designing optimal low-order digital controllers using direct parameter optimization. The controller can be of any order and structure, e.g., proportional-integral-derivative controllers are considered a subset of the controllers defined in this formulation. Advantages of this method are primarily a result, first, of its closeness to the classical s-domain techniques where continuous-time designers have gained valuable design experiences. Second, the solution algorithm is based on efficient numerical gradient search methods developed for continuous-time systems.[9]

This paper is organized as follows. In Sec. II, the control problem with computation time delay is defined. Section III presents a state–space formulation of the problem in the w domain and some interesting remarks. In Sec. IV, the optimal W-synthesis method is applied to the solution of an optimal design. Results applied to the benchmark problem of a two-mass-spring model are described in Sec. V, and conclusions are summarized in Sec. VI.

II. Control Problem with Computation Time Delay

When a continuous-time plant is controlled by a discrete-time controller (which is implemented in a microprocessor), one usually introduces a digital-to-analog (D/A) converter in the control-input path and an analog-to-digital (A/D) converter in the measurement path of a single-rate sampled-data (SRSD) system. Here, we assume that the D/A converter is of a zero-order-hold type. In the treatment of such a system, one also assumes that the D/A and A/D converters are ideal converters (i.e., synchronization, no delay). In certain cases, however, the SRSD system may witness a small time delay in the feedback path of a sampled-data control system. The time delay is the result of a finite amount of time needed to update the control feedback signals from a recent sampling of the measurements. Such a problem seldom occurs in a continuous-time (i.e., analog) controlled system. In this paper, we consider the case where the computation time delay is less than the sampling time T. One way to alleviate the problem associated with computation time delay is in the implementation of the control algorithms with a pipeline structure on fast digital processors. This design solution may be impractical and overly costly. We propose a solution that involves the redesign of the controller to directly account for the effects of the known computation time delay.

Consider a linear time-invariant system given in the following state–space description:

$$\Sigma_s: \begin{cases} \dot{x}(t) = Ax(t) + B_1 w_c(t) + B_2 u(t) \\ z(t) = C_1 x(t) + D_{12} u(t) \\ y_k = H_d x_k + D_2 v_k \end{cases} \tag{1}$$

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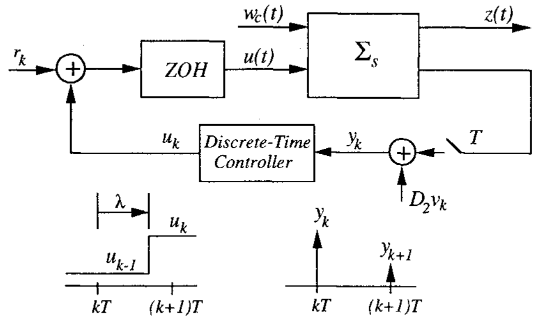

Fig. 1 Sampled-data system with computation time delay λ.

where $x(t)$ is a state vector in $\Re^{n\times 1}$, $u(t)$ a control input vector in $\Re^{m\times 1}$, and $z(t)$ a controlled output vector in $\Re^{q\times 1}$. Usually y_k, x_k, u_k, and v_k denote the responses of $y(t), x(t), u(t)$, and $v(t)$ at time $t = kT$, respectively. The output y_k is a discrete-time measurement vector in $\Re^{p\times 1}$, and $w_c(t)$ is a continuous-time process disturbance vector in $\Re^{d_1\times 1}$ and v_k is a discrete-time measurement noise vector in $\Re^{d_2\times 1}$. The matrices A, B_1, B_2, C_1, D_{12}, H_d, and D_2 are constant matrices with appropriate dimensions. Also, we assume that the pair $\{A, B_2\}$ is stabilizable, $\{A, B_1\}$ is disturbable, the pairs $\{A, C_1\}$ and $\{A, H_d\}$ are both detectable. Also, the sampling time T has been a priori selected by the designers such that there is no loss in the overall system controllability and observability.

When the computation time delay is incorporated into the control synthesis, feedback control action can be represented by

$$u(t) = \begin{cases} u_{k-1}: kT \le t < kT + \lambda \\ u_k: kT + \lambda \le t < (k+1)T \end{cases} \tag{2}$$

where λ is the computation time delay (assumed known) and $\lambda < T$. A general block-diagram of the sampled-data system with the computation time-delay modeled in Eq. (2) is depicted in Fig. 1.

Let us consider the following quadratic cost function J, which defines the performance objective of the SRSD system in Eq. (1):

$$J = \lim_{t_f\to\infty} \frac{1}{2t_f}\int_0^{t_f} E\{z(t)^T Q z(t) + u(t)^T R u(t)\}\,\mathrm{d}t \tag{3}$$

where $E\{\cdot\}$ denotes the expectation operator and $z^T(t)$ is the transpose of $z(t)$. The matrices Q (and R) are symmetric positive (semi)definite weighting matrices. Our problem formulation involves a measurement-based feedback system with a low-order dynamic controller of the form

$$\begin{aligned} x_{c_{k+1}} &= A_c x_{c_k} + B_c y_k \\ u_k &= C_c x_{c_k} + D_c y_k \end{aligned} \tag{4}$$

where the controller is shift invariant and having an arbitrary structure and order r. For convenience, one can express the controller matrices in a compact notation through the single gain matrix K_y as follows:

$$K_y \equiv \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix} \tag{5}$$

The problem addressed reduces, therefore, to the design of an optimal discrete-time controller, as given in Eq. (4), that minimizes the quadratic cost function J in Eq. (3) for a sampled-data system with computation time delay in the control path, as defined in Eq. (2). The solution approach differs from the conventional z-domain approach of Ref. 2. It is based on a method in the w domain called optimal W synthesis that was developed in Ref. 8.

III. Formulation of the Synthesis Problem in the w Domain

References 7 and 8 gave a synthesis method for designing a digital controller in the w domain that is quite different from the conventional z-domain method. The method has the advantages that frequency loop shaping[10] can be generated using classical design experiences developed in the Laplace s domain, and the immediate utilization of existing optimization software for continuous-time controller designs.[9,11] The proposed method is not an emulation method; rather it is an exact formulation of the control problems in the z domain by transforming them into the w domain for control-law synthesis. In digital control systems, there are many possible sources of time delay. It is important to examine the effects of computation time delay represented in Eq. (2) on the design performance and robustness. It is expected that such a time delay can severally affect the performance and robustness of a sampled-data control system. In this paper, we extend the synthesis method proposed in Refs. 7 and 8 to include the case involving computation time delay as described in the previous section.

Let us begin with the definition of the w domain[12,13] used in the description of the synthesis model.

Definition III.1: The w domain is a complex frequency domain defined by a bilinear transformation in the following mapping[14]:

$$w \equiv (2/T)[(z-1)/(z+1)] \tag{6}$$

where z is the well-known z-transform operator with $z = e^{sT}$.

The w-domain design approach offers many useful insights into the synthesis of digital control systems particularly from the frequency-domain point of view. Some of these properties are given in Ref. 7. Using these properties, we can restate the discrete-time control problem in terms of the following equivalent problem: Find a low-order discrete-time controller that minimizes a quadratic performance index described directly in the w domain and subjected to the system state constraints described in the w domain.

For a compact description of the SRSD system in Eq. (1) at $t = kT$, let us introduce the following definitions.

Definition III.2:

$$\begin{aligned} \Phi(t) &\equiv e^{At}, \qquad \Psi(t) \equiv \int_0^t e^{A\theta}\,\mathrm{d}\theta B_2 \\ G_d\xi_k &\equiv \int_0^T \Phi(T-\theta)B_1 w_c(kT+\theta)\,\mathrm{d}\theta \end{aligned} \tag{7}$$

where $w_c(t) \in L_2[0, T]$ and ξ_k is a discrete sequence vector in $\Re^{n\times 1}$ that belongs to a class of l_2. G_d is a linear transformation defined as follows:

$$G_d: L_2[0, T] \to \Re^n$$

Namely, we treat the class of L_2 disturbances $w_c(t)$ that are square integrable in the time interval $t \in [0, T]$ or Gaussian white noises.

To solve this problem, it is convenient to obtain a state–space representation of the system defined in Eq. (1). Before doing so, we further assume that the control input $u(t)$ in Eq. (1) is zero for $t < 0$. When the continuous-time system in Eq. (1) is discretized at the given sampling time T, the discretized state equation can be expressed in the following form:

$$x_{k+1} = \Phi(T)x_k + \Psi(T)u_k + G_d\xi_k \tag{8}$$

where the matrices $\Phi(T)$, $\Psi(T)$, and $G_d\xi_k$ are defined in Eq. (7). Now, the system state equation in the w domain including the computation time delay is given in the following proposition.

Proposition III.1: Let us define an augmented state vector $\chi_w(w)$ in $\Re^{(n+m)\times 1}$ and a disturbance vector $\zeta(w)$ in the w domain as

$$\chi_w(w) \equiv \begin{bmatrix} x_w(w) \\ u_I(w) \end{bmatrix}, \qquad \zeta(w) \equiv \begin{bmatrix} \xi(w) \\ v(w) \end{bmatrix} \tag{9}$$

Then, the corresponding state equation of the synthesis model can be described in the w domain as follows:

$$\Sigma_w: \begin{cases} w\chi_w(w) = A_w\chi_w(w) + B_w u(w) + G_w^\lambda \zeta(w) \\ y(w) = H_w\chi_w(w) + L_w^\lambda u(w) + G_v^\lambda \zeta(w) \end{cases} \tag{10}$$

The state vector $\chi(w)$ (i.e., the w transform of x_k), taking into account the computation time delay, becomes an output state equation expressed in terms of the state vector $\chi_w(w)$. Namely,

$$\chi(w) \equiv C_w\chi_w(w) + D_w u(w) + E_w^\lambda \zeta(w) \tag{11}$$

with

$$A_w \equiv \begin{bmatrix} (2/T)[\Phi(T)+I_n]^{-1}[\Phi(T)-I_n] & \mathbf{0} \\ \mathbf{0} & -(2/T)I_m \end{bmatrix}$$

$$B_w \equiv \begin{bmatrix} (4/T)[\Phi(T)+I_n]^{-2}\Psi(T) \\ (4/T)I_m \end{bmatrix}$$

$$G_w^\lambda \equiv \begin{bmatrix} (4/T)[\Phi(T)+I_n]^{-2}\Phi(\lambda)G_d & \mathbf{0}_{n\times d_2} \\ \mathbf{0}_{m\times n} & \mathbf{0}_{m\times d_2} \end{bmatrix}$$

$$H_w \equiv H_d\Phi(-\lambda)[I_n \quad -\Psi(\lambda)]$$

$$L_w^\lambda \equiv L_d + H_d\Phi(-\lambda)\{\Psi(\lambda) - [\Phi(T)+I_n]^{-1}\Psi(T)\} \quad (12)$$

$$G_v^\lambda \equiv \left[-H_d[\Phi(T)+I_n]^{-1}G_d \quad D_2\right]$$

$$C_w \equiv [I_n \quad \mathbf{0}_{n\times m}]$$

$$D_w \equiv -[I_n + \Phi(T)]^{-1}\Psi(T)$$

$$E_w^\lambda \equiv \left[-[I_n+\Phi(T)]^{-1}\Phi(\lambda)G_d \quad \mathbf{0}_{n\times d_2}\right]$$

where the matrices $\Phi(T)$, $\Psi(T)$, and $G_d\xi(w)$ are defined in Eq. (7). Also, I_n denotes an identity matrix of dimension n and $\mathbf{0}_{l\times m}$ a zero matrix of dimension $l\times m$. Note that $\Phi(-\lambda)$ is nonsingular for any $\lambda\in[0,\ T)$.

Proof: See Appendix A.

We have the following interesting observations related to a sampled-data system with computation time delay.

Remark III.1: In the w domain, the system in Eq. (12) has m eigenvalues at $w=-2/T$ because of the additional term involving a delayed control input.

Remark III.2: As the computation time delay λ approaches zero, the system in Eq. (12) becomes decoupled and reduces simply to the ordinary system without the time delay λ since $\lim_{\lambda\to 0}\Phi(\lambda)=I_n$ and $\lim_{\lambda\to 0}\Psi(\lambda)=\mathbf{0}$.

Remark III.3: The computation time delay λ affects the process noise term in G_w^λ in Eq. (12). Its effect does not show up in the measurement equation in Eq. (12) because the $\Phi(\lambda)$ is canceled in the derivation of equation G_v^λ as given in Eq. (12).

Remark III.4: When the computation time delay λ exists in the system given in Eq. (10), controllability and observability of the system Σ_w are preserved if and only if the system Σ_w with $\lambda=0$ is controllable and observable in the w domain.

In optimal design problems, proper selection of performance criterion is important. Here we adhere to the performance measure defined in Eq. (3) and discretize it for the sampled-data system under consideration. For the design method under consideration, it is also necessary to find an equivalent procedure to evaluate the discretized performance index in the w domain.

For convenience let us define the following matrix:

$$\tilde{\Xi}_{(t)} \equiv \int_0^t \begin{bmatrix} \Phi(s) & \Psi(s) \\ \mathbf{0}_{m\times n} & I_m \end{bmatrix}^T \begin{bmatrix} \bar{Q} & \bar{M} \\ \bar{M}^T & \bar{R} \end{bmatrix} \begin{bmatrix} \Phi(s) & \Psi(s) \\ \mathbf{0}_{m\times n} & I_m \end{bmatrix} \mathrm{d}s \quad (13)$$

where

$$\bar{Q} = C_1^T Q C_1, \qquad \bar{M} = C_1^T Q D_{12}, \qquad \bar{R} = D_{12}^T Q D_{12} + R \quad (14)$$

Note that the matrices Q and R are defined in Eq. (3).

Proposition III.2: An exact formulation of the discretized performance criterion for the synthesis model in Eq. (10) is as follows:

$$\tilde{J}(T,\lambda) \equiv \frac{1}{2\pi}\int_0^\infty E\{z_w^T(-j\nu)\cdot\tilde{\Xi}_{(T)}\cdot z_w(j\nu)\}\frac{4/T}{(2/T)^2+\nu^2}\,\mathrm{d}\nu \quad (15)$$

where ν represents the frequency (in radians per second) along the imaginary axis in the w plane (i.e., $w=j\nu$). The ficticious criterion variables $z_w(w)$ are defined as

$$z_w(w) \equiv \begin{bmatrix} I_n & \mathbf{0}_{n\times m} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\chi_w(w) + \begin{bmatrix} -[\Phi(T)+I_n]^{-1}\Psi(T) \\ I_m \end{bmatrix}u(w) + \begin{bmatrix} -[\Phi(T)+I_n]^{-1}\Phi(\lambda)G_d & \mathbf{0}_{n\times d_2} \\ \mathbf{0}_{m\times d_1} & \mathbf{0}_{m\times d_2} \end{bmatrix}\zeta(w)$$

The weighting matrix $\tilde{\Xi}_{(T)}$ in $\tilde{J}(T,\lambda)$ is defined in Eq. (13).

Proof: See Appendix B.

An interesting property of $\tilde{J}(T,\lambda)$ is given in the following.

Remark III.5: As the sampling time T (and hence λ) approaches zero, the performance index $\tilde{J}(T,\lambda)$ becomes

$$\lim_{T\to 0}\frac{\tilde{J}(T,\lambda)}{T^2} = J$$

where J is the performance criterion defined for the continuous-time system as given in Eq. (3).

Proof: As the sampling time T approaches zero, we note that[7]

$$\begin{aligned} \lim_{T\to 0}\{(2/T)[\Phi(T)+I_n]^{-1}[\Phi(T)-I_n]\} &= A \\ \lim_{T\to 0}\{(4/T)[\Phi(T)+I_n]^{-2}\Psi(T)\} &= B_2 \\ \lim_{T\to 0}\{(4/T)[\Phi(T)+I_n]^{-2}G_d\xi_k\} &= B_1 w_c(t) \\ \lim_{T\to 0}\{-[\Phi(T)+I_n]^{-1}\Psi(T)\} &= \mathbf{0} \end{aligned} \quad (16)$$

Furthermore, since the variable $z_w(w)$ can be decomposed into $z_w(w)^T=[\chi(w)^T u(w)^T]$, the state vector $\chi(w)$ and control $u(w)$ in the w domain reduce to their equivalent $x(s)$ and $u(s)$ in the s domain. The weighting matrices in Eq. (13) become

$$\lim_{t\to 0}\left(\frac{1}{t}\right)\tilde{\Xi}_{(t)} = \begin{bmatrix} \bar{Q} & \bar{M} \\ \bar{M}^T & \bar{R} \end{bmatrix}$$

Furthermore, the shaping filter in the performance criterion $\tilde{J}(T,\lambda)$ reduces to unity as T approaches zero. Namely,

$$\lim_{T\to 0}\frac{4/T^2}{(4/T^2)+\nu^2} = 1$$

Thus, we have

$$\begin{aligned} \lim_{T\to 0}\frac{\tilde{J}(T,\lambda)}{T^2} &= \frac{1}{2\pi}\int_0^\infty E\left\{\begin{bmatrix} x(-jw_c) \\ u(-jw_c) \end{bmatrix}^T \times \begin{bmatrix} \bar{Q} & \bar{M} \\ \bar{M}^T & \bar{R} \end{bmatrix}\cdot\begin{bmatrix} x(jw_c) \\ u(jw_c) \end{bmatrix}\right\}\mathrm{d}w_c \\ &= J \end{aligned} \quad (17)$$

where w_c denotes the frequency along the imaginary axis in the s domain. Therefore, the right-hand side of Eq. (17) is equal to J, the performance criterion of the continuous-time system as given in Eq. (3).

The control problem using the optimal W-synthesis method can be stated as follows: Find a low-order discrete-time controller defined in the w domain that minimizes the performance criterion $\tilde{J}(T,\lambda)$ in Eq. (15) subjected to the state constraints given in Eqs. (10) and (11).

For practical design, the low-order controller is allowed to have an arbitrary order and structure in the w domain. For simplicity and without loss of generality, we define a new auxiliary output vector $y_\tau(w)$ as follows:

$$y_\tau(w) \equiv y(w) - L_w^\lambda u(w) \quad (18)$$

where the direct feed through term from the control input in the measurement output $y(w)$ has been removed. The controller is now formulated with the auxilliary output $y_\tau(w)$ and is given in the following state–space form:

$$wx_\tau(w) = A_\tau x_\tau(w) + B_\tau y_\tau(w)$$
$$u(w) = C_\tau x_\tau(w) + D_\tau y_\tau(w) \tag{19}$$

where $x_\tau(w)$ is the controller state vector in $\Re^{r\times 1}$. For convenience, a compact notation of the controller matrices in Eq. (19) is given by the gain matrix K_τ. Namely,

$$K_\tau \equiv \begin{bmatrix} D_\tau & C_\tau \\ B_\tau & A_\tau \end{bmatrix} \tag{20}$$

For well-posedness, the controller gain matrix D_τ must be selected such that the terms $(I_p + L_w^\lambda D_\tau)$ and $(I_m + D_\tau L_w^\lambda)$ are invertible.

To implement the controller given in Eq. (19), an equivalent controller in the z domain must be developed, and it is obtained through the inverse w mapping. The word equivalent implies that both controllers yield the same closed-loop stability and performance, one for a system described in the z domain and the other in the w domain. The relation between a dynamic compensator in the w domain and that in the discrete-time z domain was developed in Ref. 8. The result is still applicable to the case involving computation time delay. For completeness, we repeat the essential results of the transformation in the following theorem.

Theorem III.1: For a stabilizing analog-type controller in the w domain as defined in Eq. (19) and characterized by the quadruple $(A_\tau, B_\tau, C_\tau, D_\tau)$, the corresponding equivalent discrete-time controller matrices as defined in Eq. (4) are given by

$$\begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix} = \tilde{C}\tilde{A}\tilde{B} + \tilde{D} \tag{21}$$

where

$$\tilde{A} = \begin{bmatrix} [(2/T)I_r - \bar{A}]^{-1} & I_r \\ (4/T)[(2/T)I_r - \bar{A}]^{-2} & [(2/T)I_r - \bar{A}]^{-1}[(2/T)I_r + \bar{A}] \end{bmatrix}$$

$$\tilde{B} = \begin{bmatrix} \bar{B} & 0 \\ 0 & I_r \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} \bar{C} & 0 \\ 0 & I_r \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} \bar{D} & 0 \\ 0 & 0_{r\times r} \end{bmatrix}$$

and

$$\bar{A} = A_\tau - B_\tau L_w^\lambda \left(I_m + D_\tau L_w^\lambda\right)^{-1} C_\tau$$
$$\bar{B} = B_\tau \left(I_p + L_w^\lambda D_\tau\right)^{-1}$$
$$\bar{C} = \left(I_m + D_\tau L_w^\lambda\right)^{-1} C_\tau$$
$$\bar{D} = D_\tau \left(I_p + L_w^\lambda D_\tau\right)^{-1}$$

Moreover, the closed-loop eigenvalues between these two system representations are related by the mapping defined in Eq. (6). Note that inverses involved in the theorem always exist in the case of well-posed problems.

In the next section, we develop the necessary conditions for optimality in the synthesis of the controller gain matrix K_τ. The results are expressed in terms of a state–space formulation.

IV. Necessary Conditions for Optimality

Let us begin by defining a fictitious time domain (τ domain) using the following transform relation:

$$F(w) = \Im[f(\tau)] \equiv \int_{0^-}^{\infty} f(\tau)e^{-w\tau}\,\mathrm{d}\tau \tag{22}$$

where $\Im\{\mathrm{d}f(\tau)/\mathrm{d}\tau\} = wF(w) - f(0^-)$.

With the preceding definition, we can transform the state equation in the w domain as given in Eq. (12), and the performance index $\tilde{J}$ in Eq. (15) into its τ-domain equivalents. Let us define the state vector of the closed-loop system as $\chi_a^T(\tau) = [\chi_w^T(\tau), \chi_f^T(\tau), u_f^T(\tau), x_\tau^T(\tau)]$. Under a general static output-feedback law,

$$u_a(\tau) = K_\tau y_a(\tau) \tag{23}$$

where

$$y_a(\tau) = H_a^\lambda \chi_a(\tau) + G_{va}^\lambda \zeta(\tau) \tag{24}$$

We have the following closed-loop system state equation:

$$\frac{\mathrm{d}\chi_a}{\mathrm{d}\tau} = \left(A_a^\lambda + B_a^\lambda K_\tau H_a^\lambda\right)\chi_a + \left(G_{wa}^\lambda + B_a^\lambda K_\tau G_{va}^\lambda\right)\zeta(\tau) \tag{25}$$

where

$$A_a^\lambda \equiv \begin{bmatrix} A_w & 0_{(n+m)\times n} & 0 & 0 \\ (2/\sqrt{T})\,I_n & 0_{n\times m} \;\; -(2/T)\,I_n & 0 & 0 \\ 0 & 0 \quad\; 0 & -(2/T)\,I_m & 0 \\ 0 & 0 \quad\; 0 & 0 & 0_{r\times r} \end{bmatrix}$$

$$B_a^\lambda \equiv \begin{bmatrix} B_w & 0_{(n+m)\times r} \\ (2/\sqrt{T})D_w & 0 \\ (2/\sqrt{T})I_m & 0 \\ 0 & I_r \end{bmatrix}$$

$$G_{wa}^\lambda \equiv \begin{bmatrix} G_w^\lambda \\ (2/\sqrt{T})E_w^\lambda \\ 0 \\ 0_{r\times(n+d_2)} \end{bmatrix} \tag{26}$$

$$H_a^\lambda \equiv \begin{bmatrix} H_w & 0 & 0 & 0 \\ 0_{r\times(m+n)} & 0 & 0 & I_r \end{bmatrix}$$

$$G_{va}^\lambda \equiv \begin{bmatrix} G_v^\lambda \\ 0_{r\times(n+d_2)} \end{bmatrix}$$

Note that the dynamic compensator gain matrix K_τ is defined in Eq. (20) and the disturbance vector $\zeta(\tau)$ consists of white noises with covariance $E[\zeta(\tau)\zeta^T(t)] = I_{n+d_2}\delta(\tau - t)$.

Necessary conditions for optimality with respect to the controller design matrix K_τ are given in the following theorem.

Theorem IV.1: Necessary conditions for optimality are given by

$$\frac{\partial \tilde{J}}{\partial K_\tau} = B_a^{\lambda^T}\Omega\left[XH_a^{\lambda^T} + \left(G_{wa}^\lambda + B_a^\lambda K_\tau G_{va}^\lambda\right)G_{va}^{\lambda^T}\right] = 0 \tag{27}$$

where X is the steady-state state covariance satisfying the Lyapunov equation

$$\left(A_a^\lambda + B_a^\lambda K_\tau H_a^\lambda\right)X + X\left(A_a^\lambda + B_a^\lambda K_\tau H_a^\lambda\right)^T + \left(G_{wa}^\lambda + B_a^\lambda K_\tau G_{va}^\lambda\right)\left(G_{wa}^\lambda + B_a^\lambda K_\tau G_{va}^\lambda\right)^T = 0 \tag{28}$$

and Ω is the costate covariance matrix obtained from the dual Lyapunov equation

$$\left(A_a^\lambda + B_a^\lambda K_\tau H_a^\lambda\right)^T\Omega + \Omega\left(A_a^\lambda + B_a^\lambda K_\tau H_a^\lambda\right) + \Xi = 0 \tag{29}$$

The augmented weighting matrix Ξ is derived from Eq. (13), and it is given by

$$\Xi \equiv \begin{bmatrix} 0_{(n+m)\times(n+m)} & 0 & 0 \\ 0 & \Xi_{(T)} & 0 \\ 0 & 0 & 0_{r\times r} \end{bmatrix} \tag{30}$$

Furthermore, the optimum cost $\tilde{J}^o$ is

$$\tilde{J}^o = \tfrac{1}{2}\mathrm{tr}\{\Xi X\} \tag{31}$$

Note that the closed-loop system matrix $(A_a^\lambda + B_a^\lambda K_\tau H_a^\lambda)$ must be stable, i.e., its eigenvalues must be in the left-hand half of the w plane.

Proof: The proof follows immediately from the derivation in Ref. 9.

The necessary conditions are used in the numerical solution of the optimum gain matrix K_τ using a nonlinear optimization technique developed in Ref. 9. In the next section, we will apply the design method to the benchmark problem of a two-mass-spring system.

V. Design Example

A design example is used to illustrate the importance of treating directly computation time delay in the control-law synthesis. We consider the two-mass-spring (TMS) system depicted in Fig. 2. The dynamical system has four states consisting of the displacement x_1 of the mass A, its velocity x_2, the relative displacement x_3 in the spring K, and its time derivative x_4. Also, there are two control inputs, $u_1(t)$ and $u_2(t)$, and one disturbance input $w_c(t)$. To examine the effects due to computation time delay on the feedback controlled system, the optimal W-synthesis method is used to design second-order dynamic compensators for this TMS system. In the design, we assume to have four measurements consisting of two displacement variables and their time derivatives. Also the disturbance $w_c(t)$ acting on the mass B is an exponentially correlated Gaussian random process with zero mean and unity variance. It is modeled by

$$\dot{w}_c(t) = -a\, w_c(t) + \sqrt{2a}\, w_\eta(t) \tag{32}$$

where $w_\eta(t)$ is a white noise with unit power spectral density. Here, we choose the following model parameters: mass $A = 1$ kg, mass $B = 0.1$ kg, spring constant $K = (8\pi)^2/11$ N/m and $a = 50\pi$, and a sampling frequency of 40 Hz (i.e., the ratio of mass A to mass B is 10). The synthesis model of this dynamical system is in the form of Eq. (1) with two control inputs, one continuous-time disturbance, input, and four discrete-time measurement outputs. A mathematical model of the TMS system in state–space description is given in Table 1. For the LQG design, we choose discrete-time covariances of the measurement noises to be 0.4 for the two displacements and 0.12 for their velocities. To improve the robustness at the control path, we include the control-distribution matrix B_2 into the second and the third column of the disturbance matrix B_1, where fictitious white noises with a power spectral density of 0.1 are introduced to each control input.

In this paper, we illustrate the importance of including computation time delay into the control-law synthesis with two different second-order controller designs. One controller is designed assuming that the sampled-data system has no time delay (i.e., $\lambda = 0$); this corresponds to the case labeled NDY. The other controller for the design case labeled DYC corresponds to a computation time delay of $\lambda = 0.6T$, i.e., 60% of the sampling period T is accounted for in the control synthesis. Both design cases are obtained using the same quadratic performance index J described in Eq. (3) where $Q = \text{diag}[(10, 400, 1, 1)]$, $R = \text{diag}[(0.014, 0.1)]$, and a criterion vector $z(t)$ consisting of two displacements and their velocities. The optimum controller designs for these two cases are summarized in Table 2. The associated closed-loop eigenvalues are shown in Table 3, including results for the equivalent damping ratio and frequency in the s domain. Clearly, the computation time delay has degraded the damping from 0.47 to 0.21. The original closed-loop damping is nearly recovered in the case DYC. Values of the optimum cost $\bar{J}^o$ and the respective H_2 norm of the criterion and control variables [i.e., $z(t)$ and $u(t)$ considering intersample behaviors] are given in Table 4. Again, improvement in the performance is achieved in the case of DYC.

Table 5 provides the root-mean-square (rms) responses of the state and control variables $x(t)$ and $u(t)$, respectively. Improvement of the responses from the case DYN to those achieved under the case DYC is significant, especially in the velocity variables.

To illustrate effects of the time delay on time responses of the system, time responses of states and control variables to initial condition of $x_3(0) = 0.01$ m are obtained. In Fig. 3, the state responses

Table 1 State–space model of TMS system

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 57.423 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -631.655 & 0 & 10 \\ 0 & 0 & 0 & 0 & -157.080 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 17.725 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ -1 & 10 \\ 0 & 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$D_2 = \begin{bmatrix} 0.633 & 0 & 0 & 0 \\ 0 & 0.346 & 0 & 0 \\ 0 & 0 & 0.633 & 0 \\ 0 & 0 & 0 & 0.346 \end{bmatrix}$$

$$D_{12} = 0_{4\times 2}, \qquad D_{22} = 0_{4\times 2}$$

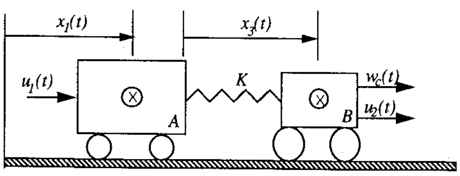

Fig. 2 TMS system.

Table 2 Optimal dynamic compensators in the z domain

Design cases	K_y (second-order dynamic compensator)					
NDY	−0.370	−2.321	−0.025	1.204	1.996	0.385
	−0.040	−0.122	0.004	−1.992	0.084	0.008
	−0.276	−0.742	−0.008	−0.077	0.842	−0.038
	0.720	0.019	−0.041	−0.405	−0.676	0.802
DYC	−0.381	−3.747	−0.051	0.677	1.721	0.924
	−0.037	−0.119	0.014	−1.461	−0.133	−0.873
	−0.148	−0.262	−0.002	−0.003	0.959	0.008
	0.051	−0.003	−0.022	−0.621	−0.039	0.728

Table 3 Stability analysis

Design cases	Closed-loop stability[a]		
	λ_z	ζ_{eq}	ω_{eq}
NDY	0.56439 ± 0.43022i	0.46592	29.444
	0.98625	1.0000	0.5539
	0.92721	1.0000	3.0230
	0.83398 ± 0.06398i	0.9191	7.7731
	0.01970	1.0000	157.08
DYN[b]	0.46514 ± 0.66040i	0.2177	39.227
	0.00320	1.0000	229.84
	0.84625 ± 0.10135i	0.8016	7.9752
	0.98625	1.0000	0.5538
	0.93107	1.0000	2.8568
	0.48670	1.0000	28.804
	0.01970	1.0000	157.08
DYC	0.05104	1.0000	119.00
	0.35196 ± 0.52004i	0.4304	43.243
	0.74482 ± 0.29141i	0.5139	17.389
	0.98828	1.0000	0.4717
	0.94203 ± 0.09648i	0.4711	4.6281
	0.01970	1.0000	157.08

[a]Here ζ_{eq} and ω_{eq} are s domain equivalent. [b]K_τ from NDY.

Table 4 Optimum $\tilde{J}^o$ and two norm of criterion and control variables

Design	$\tilde{J}^o$	$\|z\|_2$	$\|u\|_2$
NDY	0.0106	1.9936	6.8759
DYN	0.0169[a]	2.8287	8.2160
DYC	0.0126	2.4962	7.1911

[a]Nonoptimum.

Table 5 RMS responses of state and control variables

Variable	NDY	DYN	DYC
x_1	0.4721	0.5187	0.3012
x_2	0.0933	0.2566	0.1691
x_3	0.7297	0.9791	0.5436
x_4	1.0686	3.4808	2.0919
u_1	1.8102	2.2903	2.1106
u_2	1.4669	2.3862	1.7812

Table 6 Multivariable gain and phase margins in the control loop

Type		NDY	DYN	DYC
Robustness	GM, dB[a]	[−7.9, 4.0]	[−3.8, 2.5]	[−5.8, 3.9]
Test I	PM, deg	±34	±18	±28
Robustness	GM, dB	[−4.1, 8.2]	[−2.4, 2.7]	[−3.9, 5.9]
Test II	PM, deg	±36	±14	±29

[a]GM and PM denote gain and phase margins, respectively.

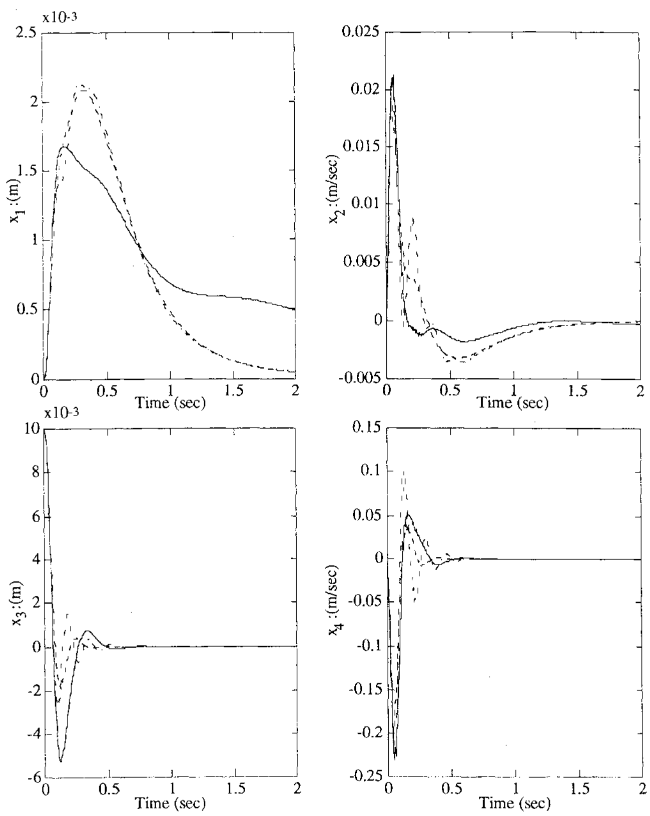

Fig. 3 State responses to an initial condition $x_3(0) = 0.01$ m: - - - -, NDY case; —·—, DYN case; and ——, DYC case.

are shown to be sensitive to the time delay. Control responses of the closed-loop system are shown in Fig. 4. The peak responses are reduced when the computation time delay is included in the design synthesis (case DYC). In particular, the control $u_2(t)$ has a peak value of 0.42 for the DYN case compared to 0.24 in the DYC case.

From the frequency responses of the inverse return difference (robustness test I) and return difference matrices (robustness test II) in the w domain, one can determine the multivariable gain and phase margins to the unstructured uncertainties in the control loop of the SRSD system. The results are shown in Table 6. Again, it is obvious that robustness is improved when knowledge of the computation time delay has been incorporated in the control synthesis.

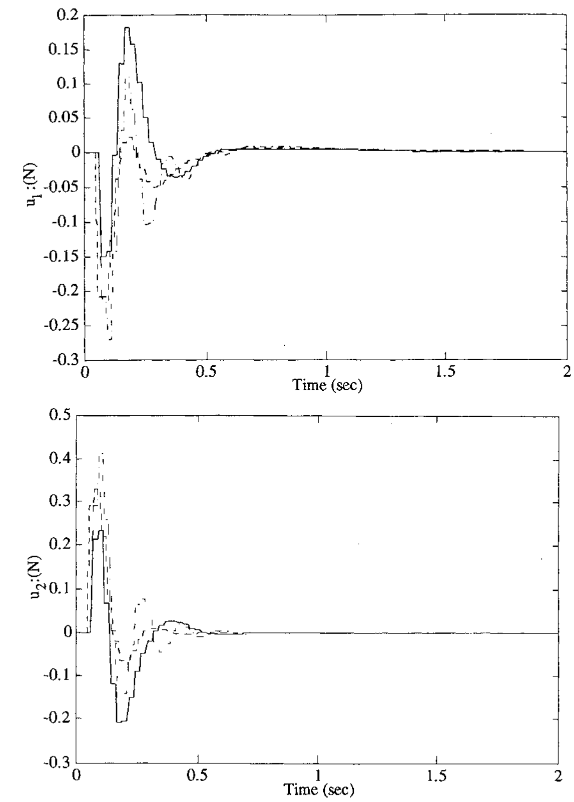

Fig. 4 Control responses to an initial condition of $x_3(0) = 0.01$ m: - - - -, NDY case; —·—, DYN case; and ——, DYC case.

VI. Conclusions

We have examined the control problem of a sampled-data system with computation time delay. The delay time is assumed to be less than the sampling time T. A complete formulation of the design problem is given. A method of including the effects of computation time delay in the control-law synthesis have been presented for a general low-order dynamic output-feedback controller of arbitrary order and structure. The optimal W-synthesis method has been used for the solution of the optimum controller design gains. The benchmark problem of a TMS system is used to illustrate the impact on stability, performance, and robustness of a sampled-data system when computation time delay is present. The proposed synthesis method allows designers to incorporate the effects of computation time delay directly into the controller design (utilizing only continuous-time techniques), and as a result enhances the overall system stability, performance, and robustness.

Appendix A: Sampled-Data System Represented in w Domain

To prove Proposition III.1, let us begin with the discretization of the continuous-time system in Eq. (1), taking into account the time delay λ represented in Eq. (2). Here, we assume that the initial condition $x(0) = x_0$ is zero and the control $u(t)$ has no action for time $t < 0$:

$$x_{k+1} = \Phi(T)x_k + \{\Psi(T) - \Psi(T-\lambda)\}u_{k-1} + \Psi(T-\lambda)u_k + G_d\xi_k \tag{A1}$$

By transformation of the delayed system as shown in Ref. 3, we can derive a simple system equation,

$$\begin{aligned} \varsigma_{k+1} &= \Phi(T)\varsigma_k + \Psi(T)u_k + G_d^\lambda w_k \\ y_k &= H_d\Phi(-\lambda)\varsigma_k - H_d\Phi(-\lambda)\Psi(\lambda)u_{k-1} + L_d u_k + D_2 v_k \end{aligned} \tag{A2}$$

where $\Phi(t)$ and $\Psi(t)$ are defined in Eq. (7) and the matrix $G_d^\lambda \xi_k$ is defined as

$$G_d^\lambda \xi_k \equiv \int_0^T \Phi(T+\lambda-s) B_1 w_c(kT+s)\, \mathrm{d}s \tag{A3}$$

where G_d^λ is a linear transformation defined as follows:

$$G_d^\lambda : L_2[0,T] \rightarrow \Re^n$$

We then apply the w transformation given in Eq. (6) to the preceding discretized system. For simplicity, let us denote the mapping function by $\aleph(\cdot)$. Thus, the state $\varsigma(w)$, control $u(w)$, output $y(w)$, and disturbances of $w(w)$ and $v(w)$ can be easily obtained by the mapping operator $\aleph$ as follows:

$$\varsigma(w) = \aleph(\varsigma_k), \qquad u(w) = \aleph(u_k), \qquad y(w) = \aleph(y_k)$$

and

$$\xi(w) = \aleph(\xi_k), \qquad v(w) = \aleph(v_k)$$

Let us define the w transform of the delayed control u_{k-1} as $u_w(w)$, i.e., $u_w(w) \equiv \aleph(u_{k-1})$ and introduce the variable $u_I(w)$ as

$$u_I(w) \equiv u_w(w) + u(w) \tag{A4}$$

Then we obtain a state equation of $u_I(w)$ associated with the delayed control input u_{k-1}:

$$w u_I(w) = -(2/T) u_I(w) + (4/T) u(w) \tag{A5}$$

After substituting Eq. (A5) into the equations obtained from the representation of Eq. (A2) in the w domain, we obtain the system representation in the w domain as shown in Eq. (10). From the immediate step between Eq. (A1) and Eq. (A2), the equation for $\chi(w)$ equivalent to x_k in the discrete-time domain can be expressed in terms of the state $\varsigma(w)$ by substituting Eq. (A4) into the state equation of $\chi(w)$:

$$\chi(w) = \Phi(-\lambda)\varsigma(w) + \Phi(-\lambda)\Psi(\lambda)[u(w) - u_I(w)] \tag{A6}$$

Note that $G_d^\lambda \xi_k$ in Eq. (A3) can be represented by $\Phi(\lambda) G_d \xi_k$, where $G_d \xi_k$ is defined in Eq. (7).

Following results from Ref. 8, one introduces a fictitious state $x_w(w)$ into the augmented state $\chi_w(w)$ to complete the derivation of Eqs. (10) and (11).

Appendix B: Derivation of the Cost Functional $\tilde{J}(T, \lambda)$

To prove Proposition III.2, we assume that the random white-noise process $w_c(t)$ in Eq. (1) has zero mean and covariance $E[w_c(t) w_c(\tau)^T] = I_{d_1}\delta(t-\tau)$ and the control $u(t) = 0$ for $t < 0$. Let us begin with the discretization of the performance index J in Eq. (3) and the system in Eq. (1), taking into account the time delay represented in Eq. (2). For simplicity, we introduce a hold state x_k^o for the control input u, i.e., $x_k^o \equiv u_{k-1}$. The discretization of the performance index J in Eq. (3) results in the following discrete cost function $\bar{J}$:

$$\bar{J} = \lim_{N\to\infty} \frac{1}{2NT} \sum_{k=0}^{N-1} E\left\{ \int_{kT}^{(k+1)T} z(t)^T Q z(t)\, \mathrm{d}t + x_k^{o^T} R\lambda x_k^o + u_k^T R(T-\lambda) u_k \right\} \tag{B1}$$

For simplicity, we define

$$w_{c_k}(s) \equiv \int_0^s \Phi(s-\theta) B_1 w_c(kT+\theta)\, \mathrm{d}\theta \tag{B2}$$

and

$$x_k(s) \equiv x(kT+s) \tag{B3}$$

Then,

$$x_k(s) = [\Phi(s) \quad \Psi(s)] \begin{bmatrix} x_k \\ x_k^o \end{bmatrix} + w_{c_k}(s) \qquad \forall s \in (0, \lambda]$$

$$x_k(s+\lambda) = \Phi(s)[\Phi(\lambda) \quad \Psi(\lambda)] \begin{bmatrix} x_k \\ x_k^o \end{bmatrix} + \Psi(s) u_k + w_{c_k}(s+\lambda) \qquad \forall s \in (0, T-\lambda] \tag{B4}$$

Note that $\Phi(t)$ and $\Psi(t)$ are defined in Eq. (7). By substituting Eq. (B4) into Eq. (B1), we obtain

$$\bar{J} = \lim_{N\to\infty} \frac{1}{2NT} \sum_{k=0}^{N-1} E\left\{ \begin{bmatrix} \tilde{x}_k \\ u_k \end{bmatrix}^T \begin{bmatrix} \tilde{Q} & \tilde{M} \\ \tilde{M}^T & \tilde{R} \end{bmatrix} \begin{bmatrix} \tilde{x}_k \\ u_k \end{bmatrix} \right\} + \frac{1}{2T} \mathrm{tr}\left\{ B_1 B_1^T \int_0^T \int_0^s \Phi(\theta)^T C_1^T Q C_1 \Phi(\theta)\, \mathrm{d}\theta\, \mathrm{d}s \right\} \tag{B5}$$

where

$$\tilde{x}_k \equiv \begin{bmatrix} x_k \\ x_k^o \end{bmatrix}$$

$$\begin{bmatrix} \tilde{Q} & \tilde{M} \\ \tilde{M}^T & \tilde{R} \end{bmatrix} \equiv \begin{bmatrix} I_{n+m} \\ \mathbf{0} \end{bmatrix} \Lambda(\lambda) \begin{bmatrix} I_{n+m} \\ \mathbf{0} \end{bmatrix}^T + \begin{bmatrix} \Phi(\lambda)^T & \mathbf{0} \\ \Psi(\lambda)^T & \mathbf{0} \\ \mathbf{0} & I_m \end{bmatrix} \Lambda(T-\lambda) \begin{bmatrix} \Phi(\lambda) & \Psi(\lambda) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_m \end{bmatrix} \tag{B6}$$

$$\Lambda(\theta) \equiv \int_0^\theta \begin{bmatrix} \Phi(s)^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Psi(s)^T & I_m \end{bmatrix} \Omega \begin{bmatrix} \Phi(s) & \mathbf{0} \\ \mathbf{0} & \Psi(s) \\ \mathbf{0} & I_m \end{bmatrix} \mathrm{d}s$$

$$\Omega \equiv \begin{bmatrix} C_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_1 & D_{12} \\ \mathbf{0} & \mathbf{0} & I_m \end{bmatrix}^T \begin{bmatrix} Q & Q & \mathbf{0} \\ Q & Q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & R \end{bmatrix} \begin{bmatrix} C_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_1 & D_{12} \\ \mathbf{0} & \mathbf{0} & I_m \end{bmatrix}$$

After further simplification, the performance index $\bar{J}$ becomes

$$\bar{J} = \lim_{N\to\infty} \frac{1}{2N} \sum_{k=0}^{N-1} E\left\{ \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \tilde{\Xi}_{(T)} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \right\} + \frac{1}{2\lambda} \mathrm{tr}\left\{ B_1 B_1^T \int_0^\lambda \int_0^s \Phi(T-\theta)^T C_1^T Q C_1 \Phi(T-\theta)\, \mathrm{d}\theta\, \mathrm{d}s \right\} \tag{B7}$$

Note that the weighting matrix in Eq. (B7) is defined in Eq. (13). Since the second term of $\bar{J}$ on the right-hand side of Eq. (B7) is constant, we can redefine the performance index as $\tilde{J}$ where $\tilde{J}/T = \bar{J} -$ [the second term of $\bar{J}$ on the right-hand side of Eq. (B7)]. Now, transform $\tilde{J}$ into its equivalent form in the w domain via the mapping in Eq. (6) to obtain the performance index $\tilde{J}(T, \lambda)$, as given in Eq. (15).

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